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The windeffect in the southern part of the North Sea due to a single storm and the influence of the Channel

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§1. Introduction

In this report[xx]) the effect of a storm in the southern part of the North Sea and the Channel will be studied. The southern part of the North Sea will be represented by a funnel-shaped region with a small opening angle of the order of 30°. The Channel will be represented by a similar region with an equally small opening angle, see figure 1.

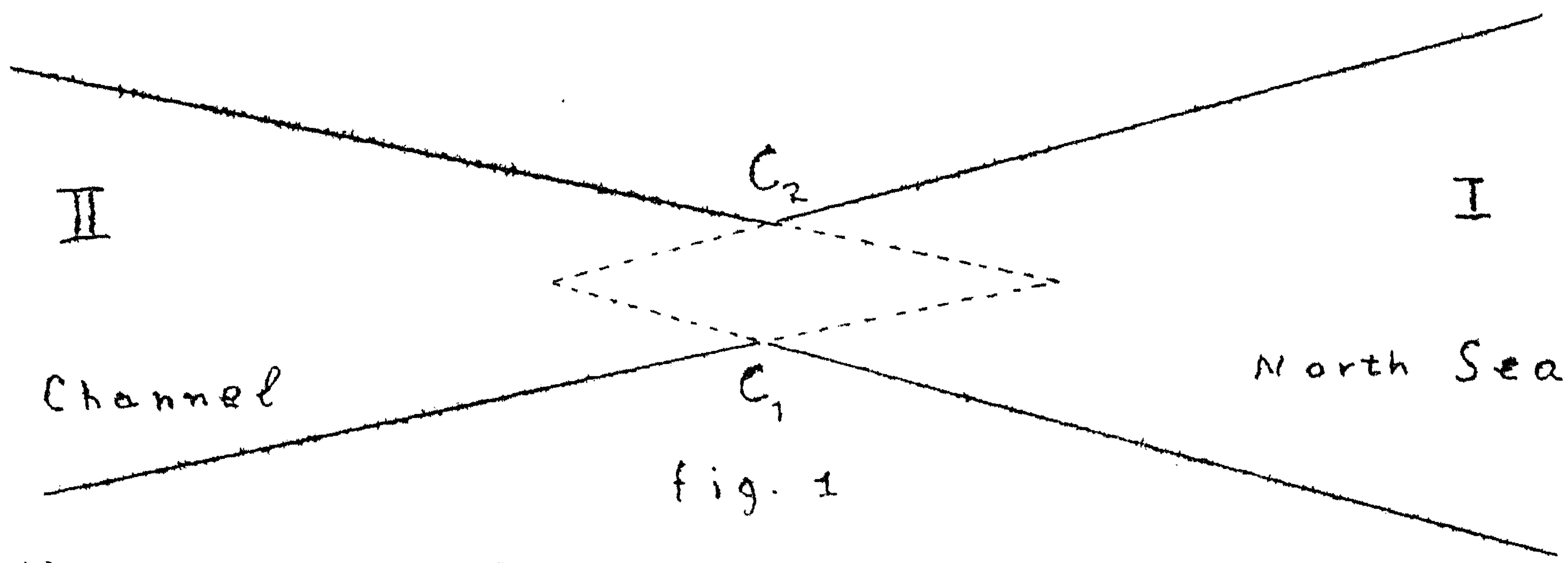

fig. 1

Both regions may be represented in polar coordinates by resp. $a < r < \infty$, $0 < \theta < \alpha$ for I and $a' < r' < \infty$, $0 < \theta' < \alpha'$ for II. Since α and α' are small the arcs $r=a$ and $r'=a'$ nearly coincide with the cross section C_1C_2, and $r=a$, $r'=a'$ may be considered as to represent the common boundary of both regions. We confine our attention to a small part of the North Sea only so that the influence of the Coriolis force may be neglected, at least for the short-term effects of a storm. The dynamical equations of the motion of the North Sea are as follows

$$\left\{\begin{aligned} &\frac{\partial u}{\partial t} + \lambda u + \frac{g}{r}\frac{\partial \zeta}{\partial \theta} = \frac{U}{\rho h} \\ &\frac{\partial v}{\partial t} + \lambda v + g\frac{\partial \zeta}{\partial r} = \frac{V}{\rho h} \\ &\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{1}{h}\frac{\partial \zeta}{\partial t} = 0 \; ; \end{aligned}\right. \qquad 1.1$$

u and v are the mean circular and radial velocity of the stream the average being taken over a vertical section, U and V are the corresponding components of the surface traction of the wind on the sea, ζ is the elevation of the sea over the zero level.

The coefficient of damping λ is taken a constant. Also the depth h is a constant. Finally g and ρ are the constant of gravity and the density.

According to Weenink*) we have for the surface traction of the wind on the sea

$$\frac{\sqrt{U^2+V^2}}{\rho} = 3.5 \times 10^{-6} \; V_s^2 \qquad 1.2$$

where V_s is the velocity of the wind above sea level.

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*) Weenink, Rapport K.N.M.I, Stormvloed 1 Febr.1953, 2e vervolg, Oct. 1955, p.61.

We shall now introduce the following units

u, v	, unit of velocity	is	c	$^{km}/hr$,
r	, unit of length	is	a	km ,
ζ	, unit of elevation	is	h	m ,
t	, unit of time	is	$^{a}/c$	hr ;

then we have

$$\begin{cases} \dfrac{\partial u}{\partial t} + \lambda u + \dfrac{1}{r}\dfrac{\partial \zeta}{\partial \theta} = U \\ \dfrac{\partial v}{\partial t} + \lambda v + \dfrac{\partial \zeta}{\partial r} = V \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} + \dfrac{1}{r}\dfrac{\partial}{\partial r}(rv) + \dfrac{\partial \zeta}{\partial t} = 0, \end{cases} \qquad 1.3$$

where

$$\sqrt{U^2+V^2} = 4.5 \times 10^{-2} \; \frac{aW^2}{gh^2} , \qquad 1.4$$

W being the velocity of the wind in $^{m}/sec$.

For the Channel region similar equations are obtained but as in the Channel no wind will be considered the wind terms are absent. If necessary the variables of the Channel region will be distinguished from those of the North Sea by a dash.

At the common section C_1C_2 we have

$$r = 1 , \quad r' = a' \qquad \begin{cases} v = v' \\ \zeta = \zeta' . \end{cases} \qquad 1.5$$

Along the Dutch coast $\theta = 0$ we have approximately

Vlissingen	$r = 2.3$
Hoek van Holland	$r = 3$
Den Helder	$r = 5$.

In the following sections the main stress will be laid upon the method rather than on the numerical results. However, by way of illustration we may consider the following numerical values which roughly correspond to actual conditions *)

$\lambda = 0.16 \text{ hr}^{-1}$	$h = 30$ m
$a = 60$ km	$\alpha = 30\%$

Thus the velocity of propagation of free waves becomes

$$c = 61.5 \; ^{km}/hr.$$

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*) Schalkwijk. A contribution to the study of storm surges on the Dutch coast. II_2.

The unit of time becomes 0.98 hr.
Formula 1.4 becomes in particular

$$\sqrt{U^2+V^2} = 2.4 \times 10^{-5} W^2. \qquad 1.6$$

In order to allow a simple mathematical treatment a radial windfield is considered so that only two independent variables r and t occur in the equations.
In § 2 also the friction λ is left out of account.
For the combination North Sea plus Channel a number of different models will be taken (see figure 2).

a a closed wedge,
b a closed wedge truncated at r=1,
c an open wedge truncated at r=1 with an ocean condition at r=1,
d a symmetric double wedge,
e a wedge plus a canal,
f an asymmetric double wedge.

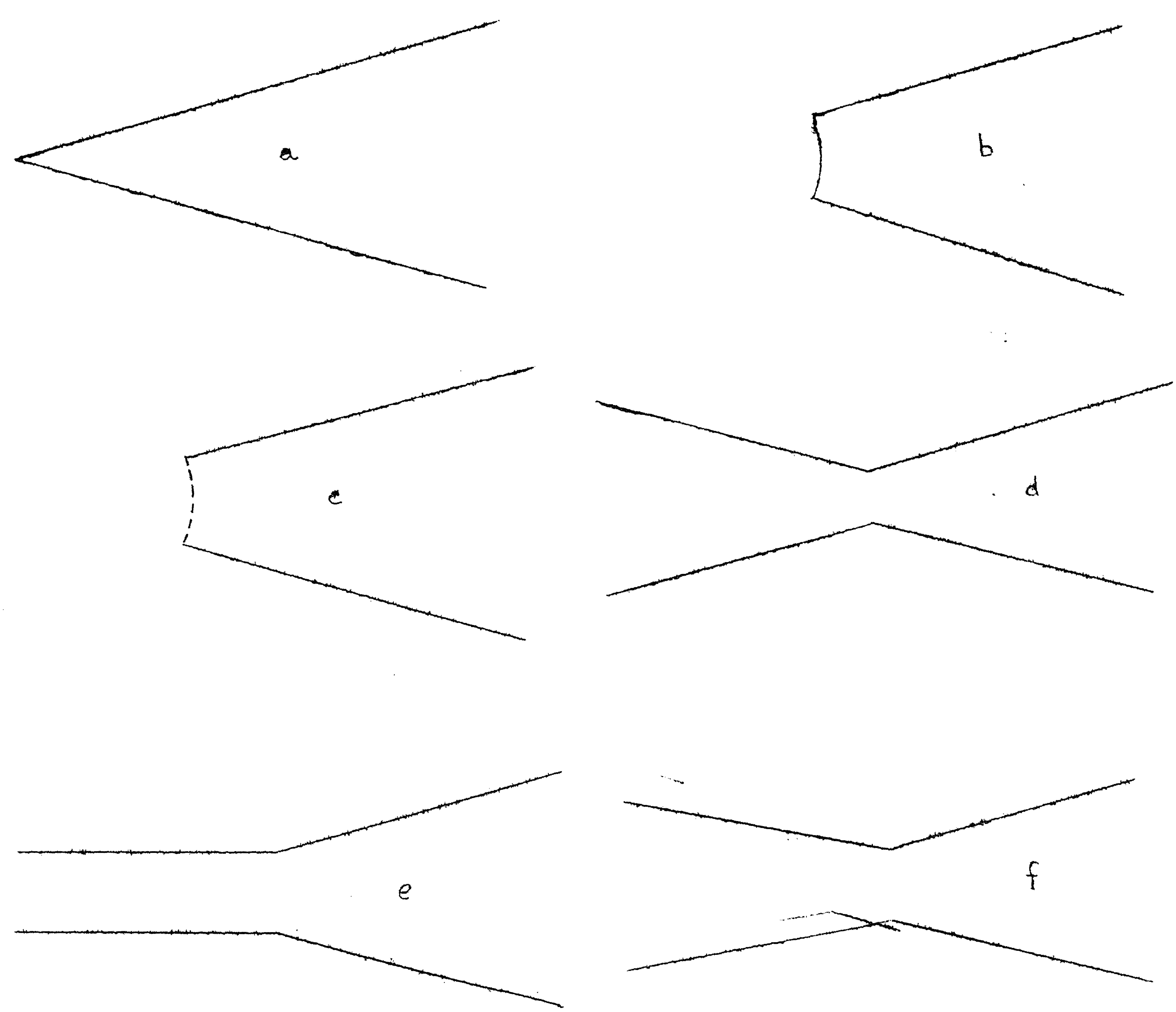


fig. 2

In § 3 the effect of a particular radial windfield viz.

$$U = 0 \qquad V = -0.01 \frac{t^2}{24^2}\left(1 - \frac{t}{48}\right) \qquad 1.7$$

(see figure 3)
upon the various models of § 2 is studied. For a,h, α the values quoted above are taken. For a number of t values the elevation ζ at r=3 (Hoek van Holland) has been computed. This has been done by means of an asymptotic expansion of ζ for large t. This expansion may be used already for $t > 24$.
From the numerical results (see figure 4) the following conclusions may be drawn

i model a and b give nearly the same elevation.
ii the effect of the ocean in model c corresponds to a reduction of the elevation ζ_b to about 25% of the original value.
iii the elevation in model d is half way between the corresponding elevation in model b and c.
iv the effect of the canal in model e is small (of the order of 10%).

In §4 the effect of the friction λ is considered. We have taken model a with $\lambda = 0.16\ hr^{-1}$. Again the elevation at r=3 for the specific windfield V has been computed. The result is given in figure 5. There is a reduction of the maximum elevation to about 40% of the original value for $\lambda = 0$. We note that in this case the maximum elevation at r=3 appears about 12 hours later than the maximum intensity of the windfield.
In § 5 an estimation is made of the influence of the Coriolis force. For model a with $\lambda = 0$ and the windfield 1.7 the effect of the Coriolis force upon the maximum elevation at the coast is almost negligible. In the numerical case the effect at r=3 (Hoek van Holland) is of the order of only 1%.
In § 6 we have considered the effect of the non-linear inertia terms. An approximate calculation has been given for model d with $\lambda = 0$ since this model most resembles the actual conditions. In order to reduce the mathematical complications a somewhat different windfield intensity is chosen (V_1 in figure 3). In order to have some compensation for the effect of the bottom friction which is left out of account here the maximum intensity of V_1 is chosen at one half of that of V. The non-linear term in the hydrodynamical equations has been considered as a perturbation term and a single iteration has been carried out.
It appears that the first-order correction ζ_1 upon the elevation ζ_0 from the linear model is always negative. Thus the non-linearity tends to lower the elevation from the linearised model (see figure 6).

§2. The influence of a radial windfield, $\lambda=0$.

We consider the sectorial region $a < r < \infty$, $0 < \theta < \alpha$. For the present we shall take $\lambda=0$. If the sea is initially at rest a uniform radial windfield

$$U = 0 \qquad V = V(t)$$

produces a θ-independent solution.
Thus from 1.3

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial \zeta}{\partial r} = V \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(rv) + \dfrac{\partial \zeta}{\partial t} = 0. \end{cases} \qquad 2.1$$

If upon 2.1 the following Laplace transformation is applied

$$\bar{\zeta}(r,p) = \int_0^\infty e^{-pt}\,\zeta(r,t)dt, \qquad 2.2$$

we obtain easily

$$\frac{d^2\bar{\zeta}}{dr^2} + \frac{1}{r}\frac{d\bar{\zeta}}{dr} - p^2\bar{\zeta} = \frac{\bar{V}}{r}. \qquad 2.3$$

Now let $\varphi(z)$ be the integer function

$$\varphi(z) = I_0(z)\int_z^\infty K_0(s)ds + K_0(z)\int_0^z I_0(s)ds, \qquad 2.4$$

for which $\quad \varphi(z) = \frac{\pi}{2} - z + \frac{\pi}{16}z^2 - \frac{1}{9}z^3 + \frac{\pi}{256}z^4 \ldots,$ $\qquad$ 2.5

and $\quad z\varphi'' + \varphi' - z\varphi = -1,$ $\qquad$ 2.6

then the solution of 2.3 is of the form

$$\bar{\zeta} = -\frac{\bar{V}}{p}\left\{\varphi(pr) + A\,K_0(pr)\right\}. \qquad 2.7$$

We shall now distinguish the following cases (see fig.2)

a a closed wedge, $a = 0$.

$$\bar{\zeta}_a = -\frac{\bar{V}}{p}\,\varphi(pr). \qquad 2.8$$

b a closed truncated wedge.
If at $r=1$ $\quad v=0$,

$$\bar{\zeta}_b = -\frac{\bar{V}}{p}\left\{\varphi(pr) - \frac{\varphi'(p)+1}{K_0'(p)}K_0(pr)\right\}. \qquad 2.9$$

c an open truncated wedge with an ocean condition.
If at $r=1$ $\quad \zeta=0$,

$$\bar{\zeta}_c = -\frac{\bar{V}}{p}\left\{\varphi(pr) - \frac{\varphi(p)}{K_0(p)}K_0(pr)\right\}. \qquad 2.10$$

Next we shall consider the combination North Sea + Channel. If the windfield is absent in the Channel we have respectively

North Sea $\quad \bar{\zeta} = -\frac{\bar{V}}{p}\left\{\varphi(pr) - A\,K_o(pr)\right\}$,

Channel $\quad \bar{\zeta}' = \frac{\bar{V}}{p}\,B\,K_o(pr')$.

The conditions 1.5 give

$$\varphi(p) - A\,K_o(p) = -\,B\,K_o(a'p)$$

$$1 + \varphi'(p) - A\,K_o'(p) = B\,K_o'(a'p),$$

so that $$A = \frac{\varphi(p)K_o'(a'p) + \{1+\varphi'(p)\}\,K_o(a'p)}{K_o(p)K_o'(a'p) + K_o'(p)\,K_o(a'p)} . \qquad 2.11$$

The following cases will be considered.

$$a'=1 , \qquad A = \tfrac{1}{2}\left\{\frac{1+\varphi'(p)}{K_o'(p)} + \frac{\varphi(p)}{K_o(p)}\right\} . \qquad 2.12$$

$$a'=\infty , \qquad A = \frac{\varphi(p)-1-\varphi'(p)}{K_o(p)-K_o'(p)} . \qquad 2.13$$

$a' \backsim 1$

In this case we may replace $K_o(a'p)$ and $K_o'(a'p)$ approximately by $K_o(p)$ and $\frac{1}{a'}K_o'(p)$ respectively. This gives

$$A = \frac{a}{a+a'}\cdot\frac{\varphi(p)}{K_o(p)} + \frac{a'}{a+a'}\,\frac{1+\varphi'(p)}{K_o'(p)} . \qquad 2.14$$

For the elevation we have accordingly

<u>d</u> $\quad a'=1.$

$$\bar{\zeta}_d = \tfrac{1}{2}\,(\bar{\zeta}_b + \bar{\zeta}_c). \qquad 2.15$$

<u>e</u> $\quad a'=\infty$.

$$\bar{\zeta}_e = -\frac{\bar{V}}{p}\left\{\varphi(pr) - \frac{\varphi(p)-1-\varphi'(p)}{K_o(p)-K_o'(p)}\,K_o(pr)\right\} . \qquad 2.16$$

<u>f</u> $\quad a' \backsim 1.$

$$\bar{\zeta}_f = \frac{a'}{a+a'}\,\bar{\zeta}_b + \frac{a}{a+a'}\,\bar{\zeta}_c . \qquad 2.17$$

We end this section with the following remark.
The case <u>e</u> with a sectorial sea and a Channel in the form of a canal with parallel sides gives the same solution as a sectorial sea in the form of an open truncated wedge alone with the following boundary condition applied at the open Channel end

$$r = 1 \qquad v = -\zeta . \qquad 2.18$$

This follows easily from 2.7 if the condition 2.18 i.e. $\frac{d\bar{\zeta}}{dr} - p\bar{\zeta} = \bar{V}$ is applied there.

§3. A particular numerical case. $\lambda=0$.

We shall consider the following model of a windfield

$$U = 0 \qquad V = -0.01\, T^2(1-\tfrac{T}{2}) \qquad 3.1$$

where $T = t/24$ (see figure 3).
For the illustrative example of the introduction this corresponds to a storm which extends over two days and which has a maximum intensity of 16 m/sec. This maximum is reached for t=32. We are interested in the value of $\zeta(r,t)$ at r=3 and $24 < t < 48$. It will become clear that under these conditions the original ζ may be obtained from the expansion of $\bar{\zeta}$, its Laplace transform for small p.
The various cases of the previous section will be considered next.

a From 2.8 and 2.5 we obtain

$$\bar{\zeta}_a = -\frac{\bar{V}}{p}\left\{\frac{\pi}{2} - pr + O(p^2)\right\}, \qquad 3.2$$

so that approximately for large t

$$\zeta_a(r,t) = -\frac{\pi}{2}\int_0^t V(\tau)d\tau + r\,V(t).$$

If 3.1 is substituted into this expression we find

$$10^2\ \zeta_a(r,t) = 4\pi\, T^3(1-\tfrac{3}{8}T) - r\,T^2(1-\tfrac{1}{2}T). \qquad 3.3$$

For r=3 we obtain for the elevation (see figure 4)

T = 1.0	ζ_a = 0.064 h
1.2	0.102
1.4	0.146
1.6	0.191
1.8	0.228
2.0	0.251
2.2	0.249

Thus we have a shift in the extremum of the elevation with respect to the extremum of the storm of about 8 units of time.

b From 2.9 we obtain

$$\zeta_b = \bar{\zeta}_a + \frac{\bar{V}}{p}\,\frac{\varphi'(p)+1}{K_o'(p)}\,K_o(pr)$$

$$= \bar{\zeta}_a + p\,\bar{V}\left\{\frac{\pi}{2}\left(\ln\frac{pr}{2}+\gamma\right) + O(p\ln p)\right\}. \qquad 3.4$$

In view of

$$\frac{\gamma + \ln p}{p^2} \doteqdot -t\ln t + t$$

$$\frac{\gamma + \ln p}{p^3} \doteqdot -\tfrac{1}{2}t^2\ln t + \tfrac{3}{4}\,t^2,$$

we find for large t approximately

$$\zeta_b = \zeta_a + 4.5 \times 10^{-7}\left\{ t\ln\frac{2t}{er} - \frac{1}{32}t^2\ln\frac{2t}{e^{3/2}\,r}\right\}. \qquad 3.5$$

For $t = 24$ we find $\zeta_b - \zeta_a = 2.6 \times 10^{-4}$

$t = 48$ $\qquad \zeta_b - \zeta_a = 3.1 \times 10^{-4}$,

which shows that the contribution of the "reflection" $\dfrac{\bar{V}}{p}\dfrac{\varphi'(p)+1}{K_o'(p)}K_o(pr)$ is entirely negligible.

Thus we may safely take

$$\zeta_b(r,t) = \zeta_a(r,t). \qquad 3.6$$

c From 2.10 we obtain for small p

$$\bar{\zeta}_c = -\frac{\bar{V}}{p}\left\{-(r-1)p - \frac{\pi}{2}\ln r\,\frac{1}{\ln\frac{p}{2}+\gamma} + O\left(\frac{p}{\ln p}\right)\right\}. \qquad 3.7$$

We have the Laplace pair

$$\frac{1}{p^{m+1}}\,\frac{1-\frac{e^{\gamma}}{2}p}{\ln\frac{p}{2}+\gamma} \doteqdot -\left(\frac{e^{\gamma}}{2}\right)^m \int_{m-1}^{m}\frac{(2e^{-\gamma}t)^u}{\Gamma(u+1)}\,du. \qquad 3.8$$

For large t and t>>m we have approximately

$$-\left(\frac{e^{\gamma}}{2}\right)^m\int_{m-1}^{m}\frac{(2e^{-\gamma}t)^u}{\Gamma(u+1)}\,du \approx \frac{t^{m-1}\left(\frac{m}{2}e^{\gamma}-t\right)}{m!\,\ln(2e^{-\gamma}t)}. \qquad 3.9$$

Thus we have from

$$\bar{\zeta}_c \approx (r-1)\,\bar{V} - 0.01\,\frac{\pi\ln r}{24^2}\left(\frac{1}{p^4}-\frac{1}{16p^5}\right)\frac{1-\frac{e^{\gamma}}{2}p}{\ln\frac{p}{2}+\gamma}$$

approximately for large t

$$\zeta_c(r,t) = -0.01\,T^2\left\{(r-1)\left(1-\frac{T}{2}\right) + \ln r\,\frac{1.50-13.3\,T+4.72\,T^2}{\ln T + 3.29}\right\}. \qquad 3.10$$

For $r = 3$ we obtain for ζ_c (see figure 4)

$T = 1.0$	$\zeta_c = 0.015\ h$
1.2	0.024
1.4	0.035
1.6	0.048
1.8	0.059
2.0	0.068
2.2	0.074

Thus values are obtained which are about 1/4 of those obtained in the case of a closed sectorial sea.

d According to 2.14 we have

$$\zeta_d(r,t) = \tfrac{1}{2} \left\{ \zeta_b(r,t) + \zeta_c(r,t) \right\}, \qquad 3.11$$

so that from a and c the following table may be derived (see figure 4)

$T = 1.0$	$\zeta_d = 0.040\ h$
1.2	0.063
1.4	0.091
1.6	0.120
1.8	0.144
2.0	0.160
2.2	0.162

e According to 2.15 we have for small p approximately

$$\bar{\zeta}_e = \bar{\zeta}_a + \frac{\pi/2\ K_o(pr)}{1-p \ln \frac{e^\gamma}{2} p} \bar{V}. \qquad 3.12$$

Since for $0 < p < 1$ the term $-p \ln \frac{e^\gamma}{2} p$ has its maximum $2e^{-1-\gamma} = 0.41$ at $p=2e^{-1-\gamma} = 0.41$, only a small error is made if we take

$$\bar{\zeta}_e = \bar{\zeta}_a - \frac{\pi}{2} \left(\ln \frac{pr}{2} + \gamma \right) \bar{V}. \qquad 3.13$$

The original can be found easily. We need the following pairs

$$\frac{\gamma + \ln p}{p^3} \doteqdot -\frac{1}{2} t^2 \ln t + \frac{3}{4} t^2,$$

$$\frac{\gamma + \ln p}{p^4} \doteqdot -\frac{1}{6} t^3 \ln t + \frac{11}{36} t^3.$$

We find

$$\zeta_e = \zeta_a + 0.0157\ T^2 \left\{ \left(1 - \frac{T}{2}\right) \ln \frac{r}{48T} + \left(\frac{3}{2} - \frac{11}{12} T\right) \right\} \qquad 3.14$$

and for $r = 3$ (see fig. 4)

$T = 1.0$	$\zeta_e = 0.051 \; h$
1.2	0.084
1.4	0.124
1.6	0.166
1.8	0.203
2.0	0.230
2.2	0.229

The following approximation of ζ_e

$$\bar{\zeta}_e = \bar{\zeta}_a + \frac{\pi}{2} \bar{V} K_o(pr) \qquad 3.15$$

admits an interesting interpretation. It represents the elevation at r in a wedge where at the origin a sink is placed of intensity

$$f = \zeta_a(0,t). \qquad 3.16$$

By a sink of intensity f is meant here

$$\lim_{r \to 0} \frac{1}{\alpha\pi} \int_0^{\alpha\pi} rv \, d\theta = - f(t). \qquad 3.17$$

This corresponds to

$$r \to 0 \qquad \bar{\zeta} \backsim p \, \bar{f} \ln r$$

so that the elevation at r may be represented by

$$\bar{\zeta} = - \frac{\bar{V}}{p} \varphi(pr) - p \, \bar{f} K_o(pr)$$

Since $\bar{\zeta}_a(0) = - \frac{\pi}{2} \frac{\bar{V}}{p}$ we have $\bar{f} = \bar{\zeta}_a(0)$ which proves the assertion.

§4. The influence of λ

We shall consider the simplest case of the elevation in a closed wedge. The equations are

$$\left\{ \begin{aligned} &\left(\frac{\partial}{\partial t} + \lambda\right) v + \frac{\partial \zeta}{\partial r} = V \\ &\frac{1}{r} \frac{\partial}{\partial r} (rv) + \frac{\partial \zeta}{\partial t} = 0, \end{aligned} \right. \qquad 4.1$$

from which

$$\frac{d^2 \bar{\zeta}}{dr^2} + \frac{1}{r} \frac{d \bar{\zeta}}{dr} - q^2 \bar{\zeta} = \frac{\bar{V}}{r}, \qquad 4.2$$

where

$$q^2 = p(p + \lambda). \qquad 4.3$$

The solution becomes for a closed wedge

$$\bar{\zeta}_a(\lambda) = -\frac{\bar{V}}{q}\ \varphi(qr). \qquad 4.4$$

For small p we have approximately

$$\bar{\zeta}_a(\lambda) = -\frac{\bar{V}}{q}\left(\frac{\pi}{2} - qr\right). \qquad 4.5$$

From the Laplace pair

$$\frac{1}{\sqrt{p(p+\lambda)}} = e^{-\frac{\lambda t}{2}}\, I_o\left(\frac{\lambda t}{2}\right),$$

we obtain for large t

$$\zeta_a(t,\lambda) = -\frac{\pi}{2}\int_0^t e^{-\frac{\lambda\tau}{2}}\, I_o\left(\frac{\lambda\tau}{2}\right) V(t-\tau)d\tau + r\,V(t). \qquad 4.6$$

For $\zeta(3,t,\lambda)$ with $\lambda = 0.16$ we find (see figure 5)

	$\lambda = 0.16$	$\lambda = 0$	reducting factor
T = 1	0.040	0.064	0.63
1.2	0.057	0.102	0.55
1.4	0.077	0.146	0.53
1.6	0.094	0.191	0.52
1.8	0.104	0.228	0.44
2.0	0.103	0.251	0.40
2.2	0.084	0.249	0.34

Thus the influence of λ becomes apparent in a reduction factor which decreases with time. The maximum elevation is reduced by the factor 0.4 approximately. The friction λ also effects a shift in the maximum (cf figure 2).

In this case the reduction factor might be represented approximately by the expression

$$0.9 \exp -\frac{\lambda t}{10}.$$

§5. The influence of the rotation of the earth

The equations 1.1 and 1.3 have been given under the assumption that the rotation of the earth has no appreciable influence on the behaviour of the sea in a small sectorial region.

If, however, the coefficient Ω of Coriolis is taken into account 1.3 should be replaced by

$$\left\{\begin{aligned} &\frac{\partial u}{\partial t} + \lambda u + \Omega v + \frac{1}{r}\frac{\partial \zeta}{\partial \theta} = U \\ &\frac{\partial v}{\partial t} + \lambda v - \Omega u + \frac{\partial \zeta}{\partial r} = V \\ \frac{1}{r}\ &\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial \zeta}{\partial t} = 0. \end{aligned}\right. \qquad 5.1$$

In the southern part of the North Sea Ω has the average value of 0.42 hr^{-1}. In order to study the effect from Ω we shall consider model a of a closed wedge with $\lambda=0$, $U=0$ and $V=V(t)$, $\alpha=30^{\circ}=\frac{\pi}{6}$ rad.

The equations 5.1 will be treated by means of the method of small perturbations. We put

$$\begin{cases} u = u_1 + \Omega u_2 \\ v = v_1 + \Omega v_2 \\ \zeta = \zeta_1 + \Omega \zeta_2 , \end{cases} \qquad 5.2$$

where u_1, v_1, ζ_1 are the solution of 5.1 with $\Omega=0$.

From § 2.a we have

$$u_1 = 0, \quad \bar{v}_1 = \frac{\bar{V}}{p}\left\{1 + \varphi'(pr)\right\}, \quad \bar{\zeta}_1 = -\frac{\bar{V}}{p}\varphi(pr),$$

or approximately (cf § 3.a)

$$u_1 = 0, \quad v_1 = \frac{\pi r V}{8}, \qquad \zeta_1 = -\frac{\pi}{2}\int_0^t V(\tau)d\tau + rV. \qquad 5.3$$

The perturbation terms satisfy

$$\begin{cases} \dfrac{\partial u_2}{\partial t} + \dfrac{1}{r}\dfrac{\partial \zeta_2}{\partial \theta} = -v_1 \\ \dfrac{\partial v_2}{\partial t} + \dfrac{\partial \zeta_2}{\partial r} = u_1 \\ \dfrac{1}{r}\dfrac{\partial u_2}{\partial \theta} + \dfrac{1}{r}\dfrac{\partial}{\partial r}(rv_2) + \dfrac{\partial \zeta_2}{\partial t} = 0, \end{cases} \qquad 5.4$$

with the boundary conditions

$$\theta = 0, \alpha \qquad u_2 = 0, \qquad (\theta, \alpha \text{ in radials}).$$

It is possible to construct a solution which approximately satisfies the equations, at least in the region of importance i.e. $r \sim 3$.

If we take

$$\begin{cases} u_2 = -\dfrac{r^3}{2}\theta(\alpha-\theta)\dfrac{d\varphi}{dt} \\ v_2 = 0 \\ \zeta_2 = r^2\left(\dfrac{\alpha}{2}-\theta\right)\varphi , \end{cases}$$

where $\varphi(t)$ is an arbitrary time function, substitution into 5.4 gives

$$\begin{cases} v_1 = r\varphi + \dfrac{r^3}{2}\theta(\alpha-\theta)\dfrac{d^2\varphi}{dt^2} \\ u_1 = 2r\left(\dfrac{\alpha}{2}-\theta\right)\varphi . \end{cases}$$

This is approximately true for

$$\varphi(t) = \frac{\pi}{8}V(t) \qquad V(t) = -0.01\frac{t^2}{24^2}\left(1-\frac{t}{48}\right).$$

If $24 < t < 40$ and $0 < r < 20$ we have

$$\frac{r^3}{2}\,\theta\,(\alpha-\theta)\left|\frac{d^2\varphi}{dt^2}\right| \Big/ r\varphi < 10^{-1},$$

$$\left|\frac{u_1}{v_1}\right| \backsim 2\left(\frac{\pi}{12}-\theta\right)$$

The term u_1 is not particularly small with respect to v_1, and certainly a better solution might be constructed, but already from this approximation we may conclude that the influence of Ω is very small. We have

$$\zeta_2 = \zeta_1 + \Omega\left\{\frac{\pi r^2 V}{8}\left(\frac{\pi}{12}-\theta\right)\right\}$$

For r=3 and $\theta=0$ we have

$$\zeta_2 = \zeta_a(3,t) + 0.04\,\Omega\,T^2\left(1-\frac{T}{2}\right)$$

where $T=t/24$.

The maximum disturbance is 0.0022 which is very small with respect to the values of ζ_a given in § 3.a.

Thus we may expect the influence of Ω upon the elevation to be of the order of 1%.

§6. The influence of the non-linearity

The equations 1.3 may be considered as linear approximations of the following non-linear equations

$$\begin{cases} u\dfrac{\partial u}{\partial r} + \dfrac{\partial u}{\partial t} + \lambda u + \dfrac{1}{r}\dfrac{\partial \zeta}{\partial \theta} = U \\ v\dfrac{\partial v}{\partial r} + \dfrac{\partial v}{\partial t} + \lambda v + \dfrac{\partial \zeta}{\partial r} = V \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} + \dfrac{1}{r}\dfrac{\partial}{\partial r}(rv) + \dfrac{\partial \zeta}{\partial t} = 0. \end{cases} \qquad 6.1$$

In order to obtain any information about the effect of the non-linear inertia terms we shall consider the model <u>d</u> of section two.

If $\lambda=0$ the equations reduce to

$$\begin{cases} v\dfrac{\partial v}{\partial r} + \dfrac{\partial v}{\partial t} + \dfrac{\partial \zeta}{\partial r} = V \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(rv) + \dfrac{\partial \zeta}{\partial t} = 0. \end{cases} \qquad 6.2$$

Provided the effect of the non-linear term is small, and this is certainly true if V is small, an approximate solution of 6.2 may be obtained by means of a single iteration starting from the solution of the linearised equations.

If ζ_o, v_o are the solution of 6.2 without the non-linear term we put

$$\zeta = \zeta_o + \zeta_1 \qquad v = v_o + v_1,$$

where ζ_1 and v_1 represent the "perturbations" from the non-linear term.

The "perturbations" satisfy the equations

$$\begin{cases} \dfrac{\partial v_1}{\partial t} + \dfrac{\partial \zeta_1}{\partial r} = - v_o \dfrac{\partial v_o}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(rv_1) + \dfrac{\partial \zeta_1}{\partial t} = 0 \end{cases} \qquad 6.3$$

in region I as well as in region II. The boundary conditions at r=1, the narrowest part of the Channel, are again the continuity of both ζ_1 and v_1.

The Laplace transform of the solution of 6.2 without the non-linear term may be approximated by (cf 2.15, 3.2, 3.6, 3.7)

$$\bar{\zeta}_o = - \frac{\bar{V}}{p}\left\{\varphi(pr) - \frac{1}{2}\,\frac{\varphi(p)K_o(pr)}{K_o(p)}\right\} \backsim - \frac{\pi \bar{V}}{4p}\left(1 - \frac{\ln r}{\ln p}\right). \qquad 6.4$$

$$\bar{v}_o = - \frac{\bar{V}}{p}\left\{-1 - \varphi'(pr) + \frac{1}{2}\,\frac{\varphi(p)K_o'(pr)}{K_o(p)}\right\} \backsim - \frac{\pi \bar{V}}{4r\,p^2 \ln p}. \qquad 6.5$$

For $\bar{V}$ we take the following formula

$$\bar{V} = \bar{V}_1 = \frac{C}{24^2}\,\frac{\ln p}{p^3}\left(1 - \frac{1}{16p}\right), \qquad 6.6$$

corresponding to the original

$$V_1 = - \frac{CT^2}{2}\left\{\left(1 - \frac{T}{2}\right)\ln 24\,T - 0.92\,(1 - 0.68T)\right\} \qquad 6.7$$

This type of windfield, very similar to that considered in section 3, may appear somewhat artificial but it has the advantage that the approximation for v becomes very simple as from

$$\bar{v}_o \backsim - \frac{\pi C}{48^2}\,\frac{1}{rp^5}\left(1 - \frac{1}{16p}\right)$$

the following original results

$$v_o \backsim \frac{-6\pi C}{r}\,T^4\left(1 - \frac{3}{10}T\right). \qquad 6.8$$

Thus the disturbance term in 6.3 becomes

$$- v_o \frac{\partial v_o}{\partial r} \backsim \frac{36\,\pi^2 C^2}{r^3}\,T^8\left(1 - \frac{3}{10}T\right)^2. \qquad 6.9$$

Next an approximative solution will be derived for the system

$$\begin{cases} p\bar{v} + \dfrac{\partial \bar{\zeta}}{\partial r} = r^{-3} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\bar{v}) + p\bar{\zeta} = 0, \end{cases} \qquad 6.10$$

holding in both regions I and II with continuity of $\bar{v}$ and $\bar{\zeta}$ at the

common arc $r=r'=1$.
Obviously we have for ζ the same formula in I and II

$$\bar{\zeta} = 2\, I_o(pr) \int_r^\infty K_o(p\varrho)\varrho^{-3}d\varrho + 2K_o(pr) \int_1^r I_o(p\varrho)\varrho^{-3}d\varrho - A K_o(pr).$$

The continuity of $\bar{v}$ at $r=1$ gives $\frac{\partial \zeta}{\partial r}=1$, so that

$$A = \frac{-1+2p\, I_o'(p) \int_1^\infty K_o(p\varrho)\varrho^{-3}d\varrho}{p\, K_o'(p)} .$$

For small p we have the fairly good approximation $A \backsim 1$, so that $\bar{\zeta}$ may be approximated by

$$\bar{\zeta} \backsim -\left(\frac{\ln pr}{r^2} + \frac{1}{2r^2}\right) - \left(1-\frac{1}{r^2}\right)\ln pr + \ln pr$$

or $\quad \bar{\zeta} \backsim -\frac{1}{2r^2}$.

Thus for large t we have the approximation

$$\bar{\zeta}_1 \backsim -\frac{18\pi^2 C^2}{r^2} T^8\left(1-\frac{3}{10}T\right)^2. \qquad 6.11$$

In the following table the non-linear disturbance ζ_1 is compared to the elevation ζ_o from the linearised system.

	$\zeta_o(r=3)$	$\zeta_1(r=3)$
T = 1.6	$0.20\ C\ 10^2$	$0.44\ C^2\ 10^3$
1.8	$0.25\ C\ 10^2$	$0.78\ C^2\ 10^3$
2.0	$0.28\ C\ 10^2$	$2.10\ C^2\ 10^3$

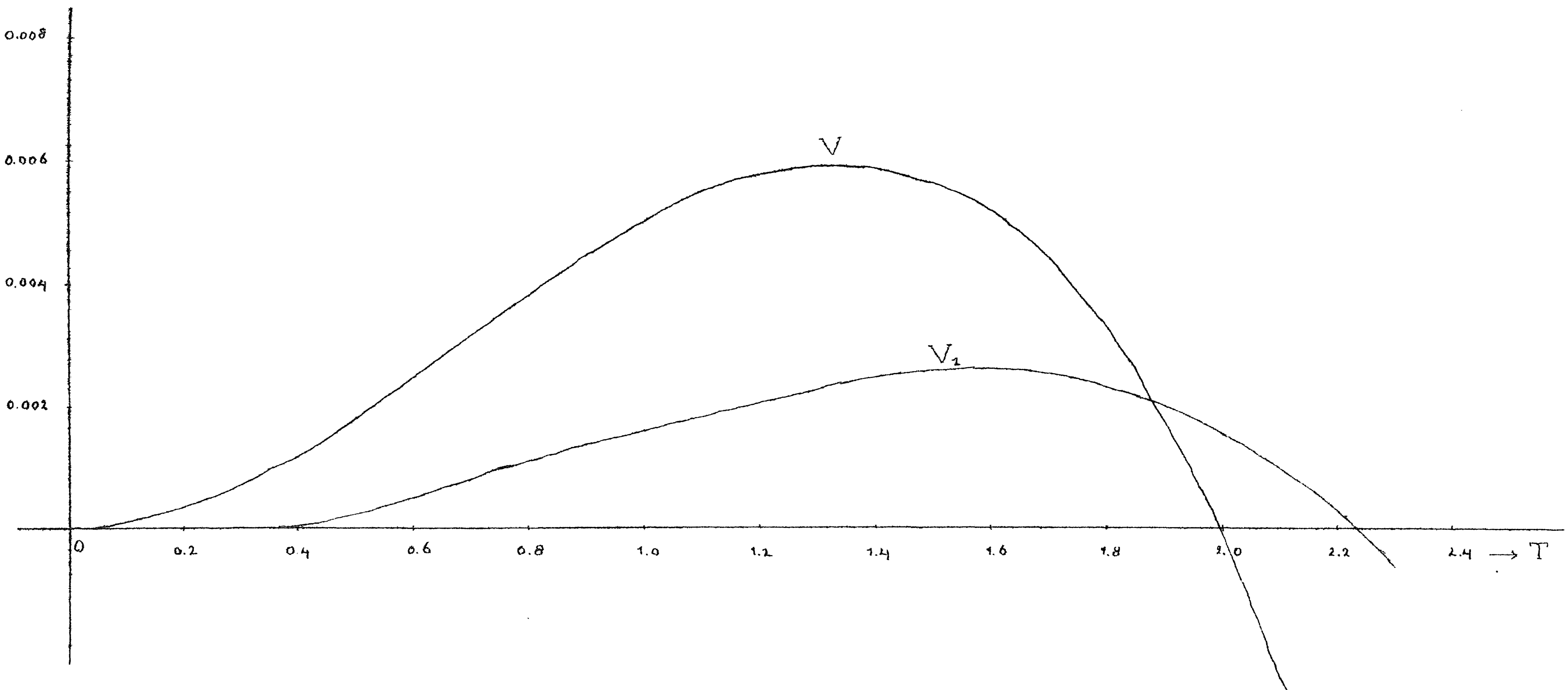
We shall take the particular value

$$C = 0.0025$$

which gives a fairly good representation of actual windfield intensity and elevation. A graph of the intensity of the windfield V_1 with this value of C has been given in fig.3. The maximum intensity of V_1 is about half that of V. This may be considered as some compensation for the influence of the friction λ which has been left out of consideration here.
Thus the windfields V and V_1 may be considered roughly aequivalent.
In the following table a picture is given of the situation at r=3 (Hoek van Holland). The linear elevation ζ_o and the first non-linear correction ζ_1 are given relative to h. The linear velocity v_o is given relative to c the velocity of propagation of free waves. Next the non-linear term $v_o \frac{\partial v_o}{\partial r}$ is compared to the linear term $\frac{\partial v_o}{\partial t}$. It appears that at the time of maximum elevation these two terms are of the same order so that the influence of the non-linearity upon the maximum

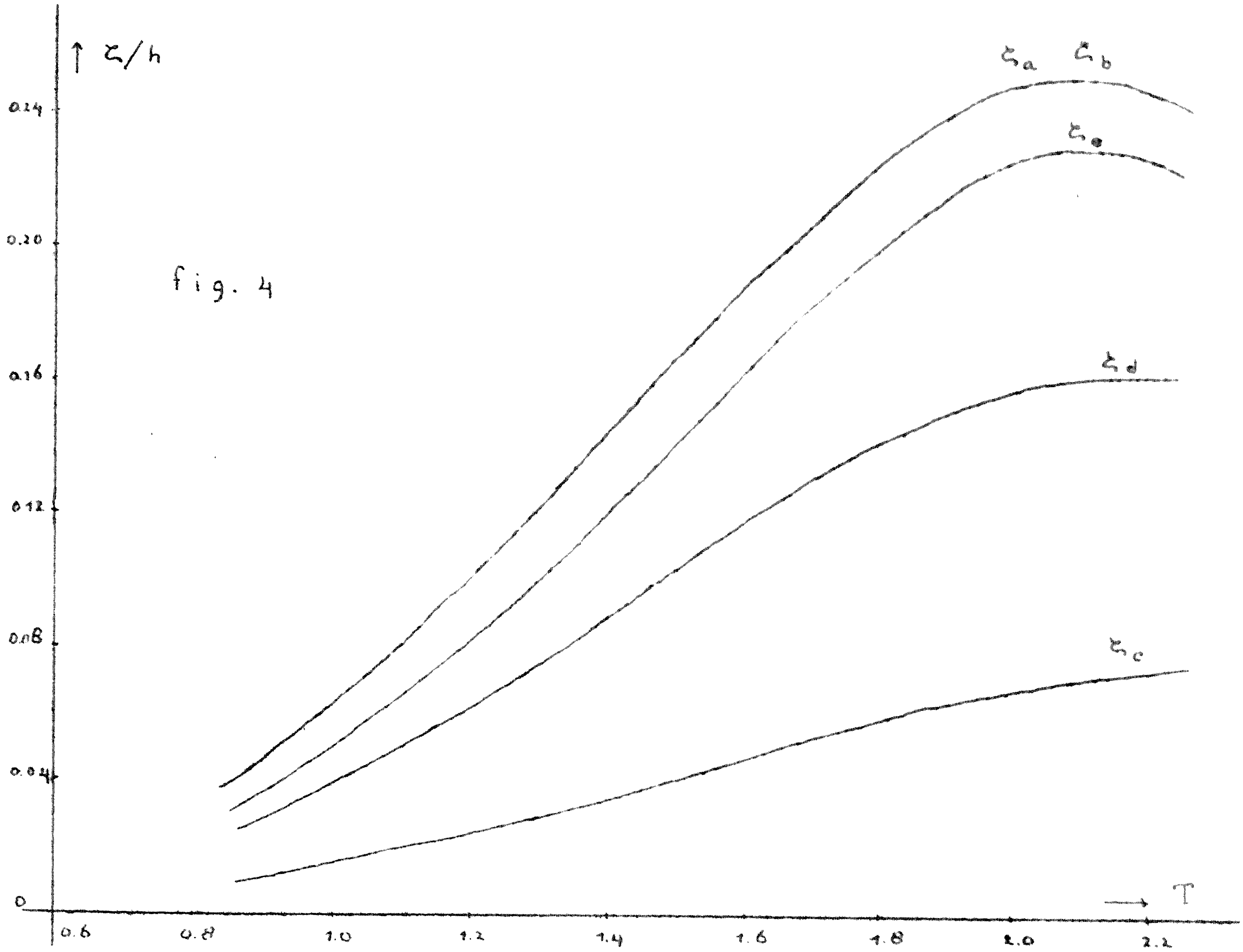
elevation of ζ_o already for this reason cannot be expected to be negligible. Fortunately the non-linearity has the effect of lowering the maximum elevation. The calculations carried out in this section are rather rough and the values obtained for ζ_1 must be considered as rough approximations only. Yet they give some idea about the quantitative effect of the non-linearity which is of the order of 5-10% (see figure 6).

T	ζ_o	ζ_1	$-v_o$	$-10^2 v_o \frac{\partial v_o}{\partial r}$	$-10^2 \frac{\partial v_o}{\partial t}$
1.0	0.016	-	0.01	-	0.16
1.2	0.025	-	0.02	0.01	0.25
1.4	0.037	-	0.03	0.03	0.34
1.6	0.050	-0.001	0.05	0.08	0.42
1.8	0.061	-0.003	0.07	0.19	0.49
2.0	0.070	-0.005	0.10	0.33	0.52
2.2	0.073	-0.008	0.12	0.52	0.49

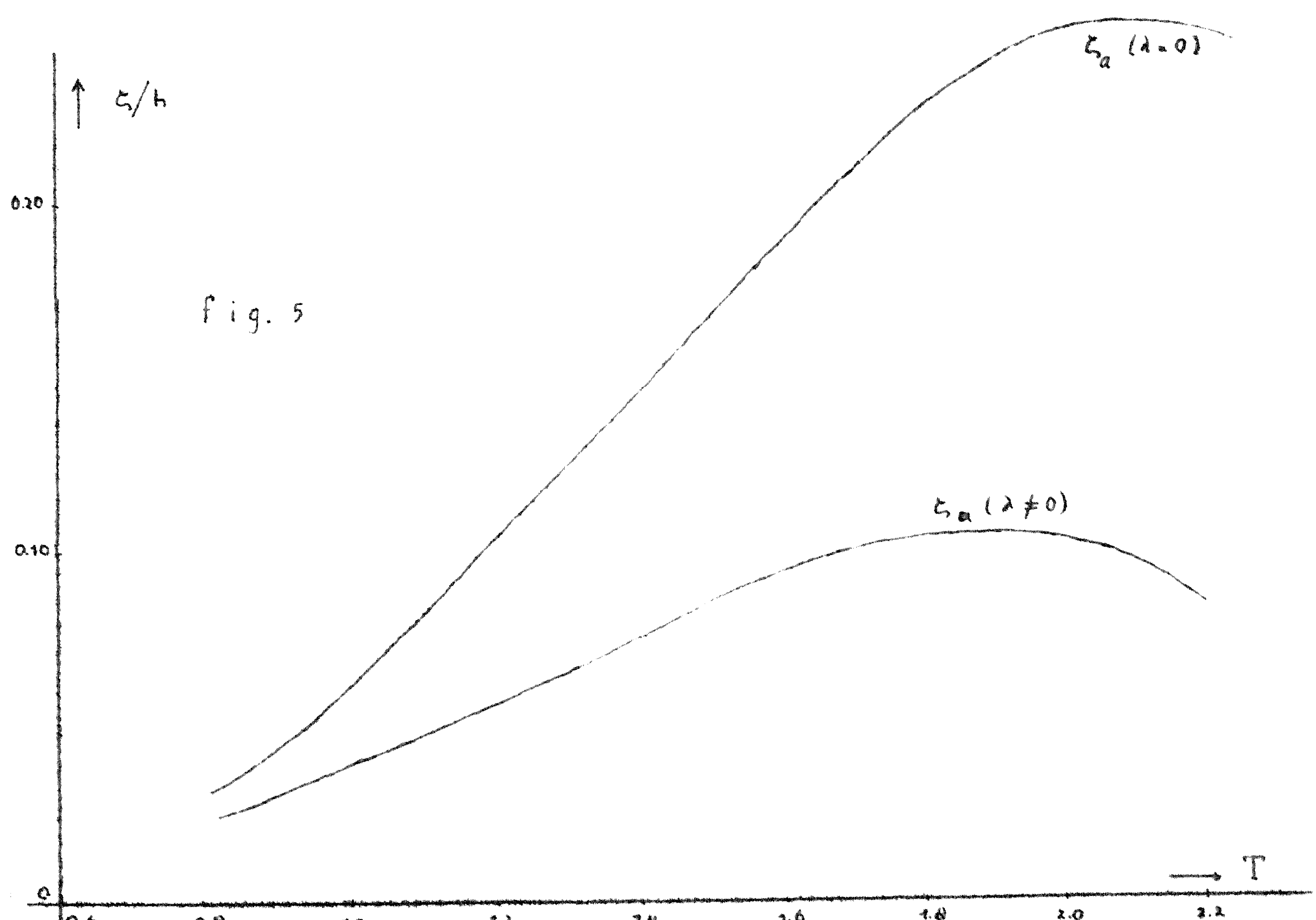


The intensity of the windfields V and V_1

fig. 3



The elevation at r=3 due to the windfield V. (λ=0).



The influence of λ upon the elevation at r=3; λ=0.16 hr^{-1}.

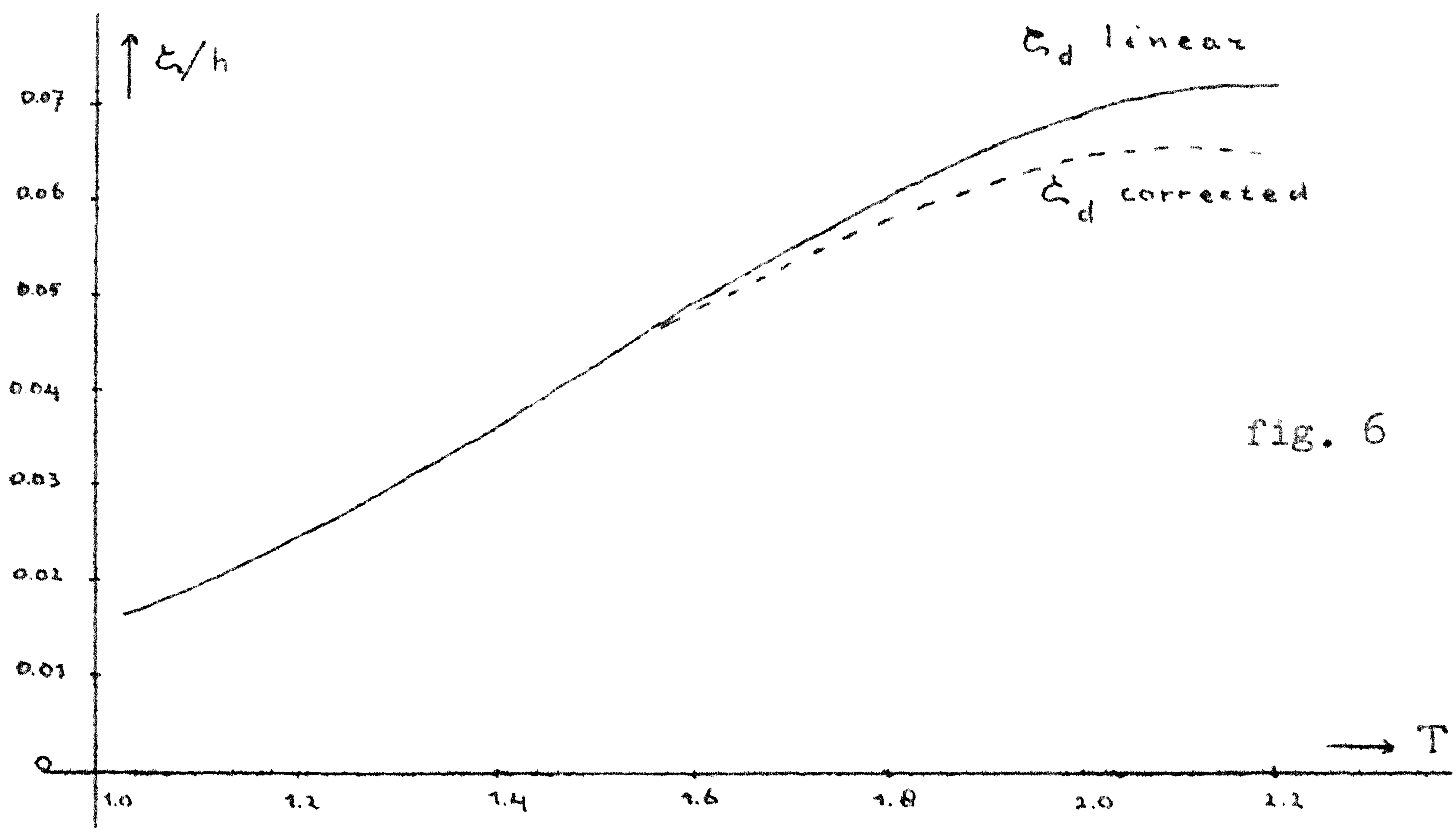


fig. 6

The influence of the non-linearity upon the elevation at $r=3$ due to windfield V_1.

	V	V_1
0	-	-
0.2	0.0004	-
0.4	12	-
0.6	25	0.0006
0.8	38	11
1.0	50	16
1.2	58	21
1.4	59	25
1.6	52	26
1.8	32	22
2.0	0	17
2.2	-	4

fig.3

	$\zeta_a = \zeta_b$	ζ_c	ζ_d	ζ_e
1.0	0.064	0.015	0.040	0.051
1.2	102	24	63	84
1.4	146	35	91	124
1.6	191	48	120	166
1.8	228	59	144	203
2.0	251	68	160	230
2.2	249	74	162	229

fig.4

	$\lambda = 0$	$\lambda \neq 0$
1.0	0.064	0.040
1.2	102	57
1.4	146	77
1.6	191	94
1.8	228	104
2.0	251	103
2.2	249	84

fig.5

	ζ_o	ζ_1
1.0	0.016	
1.2	25	
1.4	37	
1.6	50	-0.001
1.8	61	3
2.0	70	5
2.2	73	8

fig.6